Topic - Integration by Parts

Class - B.Sc. Part I

By → Dr. S.N. Sharma

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<u>Study Material</u> :→ This method of integration is applied when the integrand is of the form of product of two functions :-

$$I = \int f(x)\, g(x)\, dx$$

$$= f(x) \int g(x)\, dx - \int \left[\int g(x)\, dx\right] f'(x)\, dx$$

<u>Method</u> of selection of $f(x)$ and $g(x)$:-→

If $f(x)$ and $g(x)$ are same type of function then $g(x)$ is taken which is in the most standard form.

and Remaining is $f(x)$

Ex: → $I = \int \sec^3 x\, dx$

$= \int \sec x \sec^2 x\, dx$ Here $g(x) = \sec^2 x$, $f(x) = \sec x$

If $f(x)$ and $g(x)$ are different function then we take according to

I L A T E

1. Exponential Function
2. Trigonometric Function
3. Algebraic Function
4. Logarithmic Function
5. Inverse circular Function

Which come first from top to bottom i.e. Taken as $f(x)$ and latter as $g(x)$.

Method | STEP I → Select $f(x)$ and $g(x)$

STEP II → Find $\int g(x)\,dx$ and find $\frac{df(x)}{dx}$

STEP III → Calculate $f'(x) = \frac{d f(x)}{dx}$

then $I = \int f(x) \cdot g(x)\,dx$

$$= f(x) \int g(x)\,dx - \int \left[\int g(x)\,dx\right] f'(x)\,dx \quad \text{—(1)}$$

put above values in (1) and obtain I.

By this Rule we can integrate a Single Logarithmic function or Inverse Functions by Taking $g(x) = 1$

Problem 1. $I = \int \tan^{-1} x\,dx$

Here $f(x) = \tan^{-1} x$, $g(x) = 1$

$\frac{df(x)}{dx} = \frac{d \tan^{-1} x}{dx}$ $\quad \int g(x)\,dx = \int dx = x$

$= \frac{1}{1+x^2}$

Now Integrating by parts:

$$I = \int \tan^{-1} x\,dx$$

$$= \tan^{-1} x \int dx - \int \left[\int dx\right] \left[\frac{d}{dx} \tan^{-1} x\right] dx$$

$$= x \tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2}\,dx$$

$$= x \tan^{-1} x - \frac{1}{2} \log(1+x^2) + c$$

[Since N^r is Differential co-efficient of D^r

then integral of such function is equal to log of D^r.

Ex:→ 2. Find $\int \log x \, dx$

Let $I = \int \log x \, dx$

Here we take $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$

and $g(x) = 1 \Rightarrow \int g(x)dx = \int dx = x$

Now integrating by parts

$$I = f(x)\int g(x)\,dx - \int\left[\int g(x)dx\right]\left[\frac{d}{dx}f(x)\right]dx$$

$$= \log x \cdot x - \int x \cdot \frac{1}{x}\,dx$$

$$= x\log x - \int dx$$

$$= x\log x - x + C$$

$$= x(\log x - 1) + C$$

$$= x\left[\log x - \log e\right] + C$$

$$= x\log\frac{x}{e} + C$$

Ex:→ 3. Integrate $\int \log(1+x^2)\,dx$

Let $I = \int \log(1+x^2)\,dx$

Here we take $f(x) = \log(1+x^2) \Rightarrow f'(x) = \frac{2x}{1+x^2}$

$g(x) = 1 \Rightarrow \int g(x)dx = \int dx = x$

Now Integrating by Parts, we have

$$I = \int \log(1+x^2)\, dx$$

$$= \log(1+x^2) \int dx - \int \left[\int dx\right] \frac{d}{dx}\log(1+x^2)\, dx$$

$$= \log(1+x^2) \cdot x - \int \frac{x}{1+x^2} \cdot 2x\, dx$$

$$= x \log(1+x^2) - 2\int \frac{x^2}{1+x^2}\, dx$$

$$= x \log(1+x^2) - 2\left[\int \frac{1+x^2-1}{1+x^2}\, dx\right]$$

$$= x \log(1+x^2) - 2\left[\int dx - \int \frac{dx}{1+x^2}\right]$$

$$= x \log(1+x^2) - 2\left[x - \tan^{-1}x\right]$$

$$= x \log(1+x^2) + 2\tan^{-1}x - 2x + C$$

Ex:→ Integrate $\int \log\left[x + \sqrt{x^2+a^2}\right] dx$

Let $I = \int \log(x + \sqrt{x^2+a^2})\, dx$

Here say $f(x) = \log\left[x + \sqrt{x^2+a^2}\right]$

$g(x) = 1$

$$\Rightarrow f'(x) = \frac{1}{x+\sqrt{x^2+a^2}}\left[1 + \frac{1}{\cancel{2}\sqrt{x^2+a^2}} \cdot \cancel{2}x + 0\right]$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right]$$

$$= \frac{1}{\sqrt{x^2+a^2}}$$

$\int g(x)\, dx = \int dx = x$

Now using the Method by Parts we have.

$$I = f(x)\int g(x)\,dx - \int\left[\int g(x)\,dx\right] f'(x)\,dx$$

$$= \log\left[x+\sqrt{x^2+a^2}\right]\int dx - \int\left[\int dx\right]\frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$= x\log\left[x+\sqrt{x^2+a^2}\right] - \int\frac{x}{\sqrt{x^2+a^2}}\,dx$$

$$= x\log\left[x+\sqrt{x^2+a^2}\right] - I_1$$

Method of substitution

$$\left[\int\frac{f'(x)}{f(x)}\,dx\right.$$

put $f(x) = z$

$f'(x)\,dx = dz$

$$I_1 = \int\frac{x}{\sqrt{x^2+a^2}}\,dx$$

put $x^2+a^2 = z$

$2x\,dx = dz$

$x\,dx = \frac{1}{2}dz$

$$= \frac{1}{2}\int\frac{dz}{z^{1/2}}$$

$$= \frac{1}{2}\,\frac{z^{1/2}}{1/2} = z^{1/2} = \sqrt{x^2+a^2}$$

or

put $x = a\tan\theta$

$dx = a\sec^2\theta\,d\theta$

$x^2+a^2 = a^2\sec^2\theta$

$\sqrt{x^2+a^2} = a\sec\theta$

$\theta = \tan^{-1} x/a$

or

$$I_1 = \int\frac{x}{\sqrt{x^2+a^2}}\,dx$$

$$= \int\frac{\cancel{a}\tan\theta\; a\sec^{\cancel{2}}\theta\,d\theta}{\cancel{a}\,\cancel{\sec\theta}} = a\int\sec\theta\tan\theta\,d\theta$$

$= a\sec\theta$

$= a\cdot\frac{\sqrt{a^2+x^2}}{a}$

$= \sqrt{a^2+x^2}$

$$\therefore I = x\log\left[x+\sqrt{x^2+a^2}\right] - \sqrt{a^2+x^2}+C$$